The Discrete Laplace ("Z") Transform

Transform Definition; Properties Linear *recursion equations*, sometimes called *difference equations*, take the form

$$a_{k+m} + p_1\, a_{k+m-1} + p_2\, a_{k+m-2} + \cdots + p_m\, a_k = b_k, \;\; k = 0, 1, 2, \cdots,$$

where the p_k, $k = 1, 2, ..., m$ are known coefficients and $\left\{ b_k \,\middle|\, k = 0, 1, 2, \cdots \right\}$ is a known sequence. In most cases we will take these to be real but that is not an essential restriction. Just as the Laplace transform facilitates the solution of linear ordinary differential equations with constant coefficients, a discrete version of the transform, frequently called the "Z" transform, assists in solving these linear recursion equations as well as some other equations in discrete convolution form. Many of the formulas and relationships will be similar to those applying to the Laplace transform but there are significant differences; in particular the necessity to deal with discrete indices leads to some minor complications. There are various ways in which the transform can be defined; we will use the following.

Definition 1 *Let* $\left\{ a_k \,\middle|\, k = 0, 1, 2, \cdots \right\}$ *be a sequence of real or complex numbers satisfying a condition* $|a_k| \le M\,\gamma^k$, $k = 0, 1, 2, \cdots$, *where* M *and* γ *are positive real numbers. The discrete Laplace transform (after this we will bow to convention and call it the Z transform) of the sequence* $\{a_k\}$ *is the function* $\hat{a}(z) \equiv (\mathcal{Z}\,\{a_k\})(z)$ *of the complex variable* z *defined by the series*

$$\left(\mathcal{Z}\,\{a_k\}\right)(z) = \sum_{k=0}^{\infty} a_k\, z^{-(k+1)}.$$

Remarks The growth condition on the sequence guarantees, applying the *ratio test* for series convergence, that the series converges for all z with

$|z| > \gamma$, uniformly and absolutely for $|z| \geq \gamma_1$ if $\gamma_1 > \gamma$. This convergence property implies that $\hat{a}(z)$ is an *analytic* or *holomorphic* function of z for $|z| > \gamma$; in particular, derivatives of $\hat{a}(z)$ to any order can be computed by differentiating the series term by term.

The inverse transform, $(\mathcal{Z}^{-1}(\hat{a}(z))\,(k)$, is obtained simply by expanding the function $\hat{a}(z)$, analytic in some region $|z| > \gamma$, in powers of $\frac{1}{z}$ and then forming the sequence of coefficients of power of $\frac{1}{z}$ thus obtained.

Let us list some sample transforms. For a real or complex number r the sequence $\{a_k\}$ with $a_k = r^k$ has the transform

$$\hat{a}(z) = \sum_{k=0}^{\infty} \frac{r^k}{z^{k+1}} = \frac{1}{z} \sum_{k=0}^{\infty} \frac{r^k}{z^k} = \frac{1}{z} \frac{1}{1 - \frac{r}{z}} = \frac{1}{z - r}, \;\; |z| > r.$$

This formula is valid for both real and complex r, a fact we will make use of shortly. We should note in particular that, in contrast to the (continuous) Laplace transform, the Z-transform of the sequence with $a_k = 1$ for all k is $\frac{1}{z-1}$ while the transform of the sequence having $a_0 = 1,\; a_k = 0,\; k > 0$ has the transform $\frac{1}{z}$.

For the study of oscillatory sequences it is important to have the transforms $\left(\mathcal{Z}\left\{a^k \cos bk\right\}\right)(z)$ and $\left(\mathcal{Z}\left\{a^k \sin bk\right\}\right)(z)$ for real numbers a and b. We have, of course

$$a^k \cos bk = \frac{1}{2}\left(\left(a\, e^{ib}\right)^k + \left(a\, e^{-ib}\right)^k\right).$$

Using our earlier formula for the cases $r = a\, e^{ib}$ and $r = a\, e^{-ib}$ we see that

$$\left(\mathcal{Z}\left\{a^k \cos bk\right\}\right)(z) = \frac{1}{2}\left(\frac{1}{z - ae^{ib}} + \frac{1}{z - ae^{-ib}}\right) = \frac{z - a\,\cos b}{(z - a\,\cos b)^2 + a^2 \sin^2 b}.$$

A similar computation shows that

$$\left(\mathcal{Z}\left\{a^k\ \sin\ bk\right\}\right)(z)\ =\ \frac{a\ \sin b}{(z-a\ \cos b)^2+a^2\sin^2 b}.$$

We conclude with two simple examples; first we note that if $a_k\ =\ k$ for all k then

$$\hat{a}(z)\ =\ (\mathcal{Z}\{k\})(z)\ =\ \sum_{k=1}^{\infty}\frac{k}{z^{k+1}}\ =\ -\frac{d}{dz}\left(\sum_{k=1}^{\infty}\frac{1}{z^k}\right)$$

$$=\ (\text{with } j=k-1),\ =\ -\frac{d}{dz}\left(\sum_{j=0}^{\infty}\frac{1}{z^{j+1}}\right)\ =\ -\frac{d}{dz}\left(\frac{1}{z-1}\right)\ =\ \frac{1}{(z-1)^2}.$$

Secondly, suppose $a_k\ =\ \frac{1}{k!},\ \ k\ =\ 0,1,2,\ldots$. Then we have

$$\hat{a}(z)\ =\ \sum_{k=0}^{\infty}\frac{1}{k!z^{k+1}}\ =\ \frac{1}{z}\sum_{k=0}^{\infty}\frac{z^{-k}}{k!}\ =\ \frac{e^{-z}}{z},\ \ |z|\ >\ 0.$$

Now we will list some of the general properties of the transform and give brief verifications of those properties.

Property I: Z-transform of $\left\{\mathbf{r^{k+1}\,a_k}\right\},\ \mathbf{r\neq 0}$. Here we have

$$\left(\mathcal{Z}\left\{r^{k+1}\,a_k\right\}\right)\ =\ \sum_{k=0}^{\infty}a_k\left(\frac{r}{z}\right)^{k+1}\ =\ (\mathcal{Z}\{a_k\})\left(\frac{z}{r}\right)\ =\ \hat{a}\left(\frac{z}{r}\right).$$

Property II: The Z-transform of $\{\mathbf{a_{k+n}}\}$. Here we have

$$(\mathcal{Z}\{a_{k+n}\})(z)\ =\ \sum_{k=0}^{\infty}a_{k+n}\,z^{-(k+1)}\ =\ z^n\sum_{k=0}^{\infty}a_{k+n}\,z^{-(k+n+1)}$$

$$=z^n\sum_{k=n}^{\infty}a_k\,z^{-(k+1)}=z^n\left(\hat{a}(z)-\sum_{k=0}^{n-1}a_k\,z^{-(k+1)}\right)=z^n\,\hat{a}(z)-\sum_{k=0}^{n-1}a_k\,z^{n-(k+1)}.$$

Property III: Z-transform of $\{\mathbf{k(k-1)\cdots(k-n+1)\,a_k}\}$, $\mathbf{n}$ (integer) $>$ 0. We have

$$(\mathcal{Z}\,\{k(k-1)\cdots(k-n+1)\,a_k\})\,(z) \;=\; \sum_{k=0}^{\infty} k(k-1)\cdots(k-n+1)\,a_k\,z^{-(k+1)} \;=$$

$$\sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1)\,a_k\,z^{-(k+1)} \;=\; \left(-\frac{d}{dz}\right)^n \left(\sum_{k=n}^{\infty} \frac{a_k}{n!} z^{-(k+1-n)}\right)$$

$$=\; (-1)^n \frac{d^n}{dz^n}\left(\sum_{k=0}^{\infty} \frac{a_{k+n}}{n!} z^{-(k+1)}\right) \;=\; \frac{(-1)^n}{n!}\,\frac{d^n}{dz^n}\,(\mathcal{Z}\,\{a_{k+n}\})\,(z).$$

Then using Property II we obtain

$$(\mathcal{Z}\,\{k(k-1)\cdots(k-n+1)\,a_k\})\,(z)$$

$$=\; \frac{(-1)^n}{n!}\,\frac{d^n}{dz^n}\left(z^n\,\hat{a}(z) - \sum_{k=0}^{n-1} a_k\,z^{n-(k+1)}\right) \;=\; \frac{(-1)^n}{n!}\,\frac{d^n}{dz^n}\,(z^n\,\hat{a}(z))\,.$$

Remarks In particular, then

$$\left(\mathcal{Z}\left\{k(k-1)\cdots(k-n+1)\,r^k\right\}\right)(z) \;=\; \frac{(-1)^n}{n!}\,\frac{d^n}{dz^n}\left(z^n\,\left(\mathcal{Z}\left\{r^k\right\}\right)(z)\right)$$

$$=\; \frac{(-1)^n}{n!}\,\frac{d^n}{dz^n}\left(z^n \sum_{k=0}^{\infty} \frac{r^k}{z^{k+1}}\right) \;=\; \frac{(-1)^n}{n!}\,\frac{d^n}{dz^n}\left(\sum_{k=0}^{n-1} r^k\,z^{(n-k-1)} \;+\; \sum_{k=n}^{\infty} \frac{r^k}{z^{(k+1-n)}}\right)$$

$$=\; r^n\,\frac{(-1)^n}{n!}\,\frac{d^n}{dz^n}\left(\sum_{k=n}^{\infty} \frac{r^{k-n}}{z^{(k+1-n)}}\right) \;=\; r^n\,\frac{(-1)^n}{n!}\,\frac{d^n}{dz^n}\left(\sum_{k=0}^{\infty} \frac{r^k}{z^{k+1}}\right)$$

$$=\; r^n\,\frac{(-1)^n}{n!}\,\frac{d^n}{dz^n}\left(\frac{1}{z-r}\right) \;=\; \frac{r^n}{(z-r)^{n+1}}.$$

As an important corollary, we see that the inverse transform of $\frac{1}{(z-r)^{n+1}}$ is the sequence $\left\{\frac{1}{r^n}\,k(k-1)\cdots(k-n+1)\,r^k\right\} \;=\; \left\{k(k-1)\cdots(k-n+1)\,r^{k-n}\right\}$. For $n \;=\; 1$ we see that the transform of $\left\{k\,r^{k-1}\right\}$ is $\frac{1}{(z-r)^2}$. This agrees with our earlier example $\{a_k\} \;=\; \{k\}$ for $r \;=\; 1$.

This result also allows us to compute

$$(\mathcal{Z}\{k^n\, a_k\})(z) \;=$$

$$\frac{(-1)^n}{n!}\,\frac{d^n}{dz^n}\,(\mathcal{Z}\{a_{k+n}\})(z) \;-\; (\mathcal{Z}\{(k(k-1)\cdots(k-n+1)-k^n)a_k\})(z).$$

The right hand side involves only powers $k^m,\;\; m \,<\, n$ so we can proceed inductively. Using the formula repeatedly, when the right hand side eventually involves only $k^0 \;=\; 1$, everything has been expressed in terms of $\hat{a}(z) \;=\; (\mathcal{Z}\{a_k\})(z)$. Thus, for example,

$$(\mathcal{Z}\{k\, a_k\})(z) \;=\; -\frac{d}{dz}\,(z\,\hat{a}(z) \;-\; a_0) \;=\; -\frac{d}{dz}\,(z\,\hat{a}(z))\,.$$

With this in hand we can proceed to compute

$$\left(\mathcal{Z}\left\{k^2\, a_k\right\}\right)(z) \;=\; \frac{1}{2}\,\frac{d^2}{dz^2}\left(z^2\,\hat{a}(z) \;-\; a_0\, z \;-\; a_1\right) - \left(\mathcal{Z}\left\{\left((k^2-k)-k^2\right)a_k\right\}\right)(z)$$

$$= \frac{1}{2}\,\frac{d^2}{dz^2}\left(z^2\hat{a}(z)\right) + (\mathcal{Z}\{k\, a_k\})(z) \;=\; \frac{1}{2}\,\frac{d^2}{dz^2}\left(z^2\hat{a}(z)\right) - \frac{d}{dz}\,(z\,\hat{a}(z)) \;=\; z^2\hat{a}^{''}(z) + z\,\hat{a}^{'}(z).$$

For a positive integer n we define the (right) n-shift of the sequence $\{a_k\}$ to be the sequence $\{a_{n+,k}\}$, where

$$a_{n+,k} \;=\; \begin{cases} 0, \;\; 0 \,\le\, k \,<\, n, \\ a_{k-n}, \;\; k \,\ge\, n. \end{cases}$$

Property IV: Z-transform of the n-Shift. We compute

$$\hat{a}_{n+}(z) \;=\; (\mathcal{Z}\{a_{n+,k}\})(z) \;=\; \sum_{k=n}^{\infty} a_{k-n}\, z^{-(k+1)}$$

$$= z^{-n}\sum_{k=n}^{\infty} a_{k-n}\, z^{-(k-n+1)} \;=\; (\text{with} j \;=\; k-n) \;=\; z^{-n}\sum_{j=0}^{\infty} a_j\, z^{-(j+1)} \;=\; z^{-n}\,\hat{a}(z).$$

Definition 2 *Let* $\left\{ a_k \,\middle|\, k = 0,1,2,\cdots \right\}$, $\left\{ b_k \,\middle|\, k = 0,1,2,\cdots \right\}$ *be two sequences, indexed as indicated. The* convolution product *of these sequences is the sequence* $\left\{ c_k \,\middle|\, k = 0,1,2,\cdots \right\}$, *written* $\{c_k\} = \{a_k\} * \{b_k\}$, *where*

$$c_k = \sum_{j=0}^{k} a_j \, b_{k-j}.$$

Property V: Z-transform of a Convolution Product On the basis of the **corresponding Laplace transform result** we should expect some sort of ordinary product of Z-transforms here, and we are not disappointed.

$$(\mathcal{Z}\{\{a_k\} * \{b_k\}\})(z) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j \, b_{k-j} \right) z^{-(k+1)}$$

$$= z \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j \, b_{k-j} \right) z^{-(k-j+1)-(j+1)} = z \sum_{\ell=0}^{\infty} \sum_{j=0}^{\infty} a_j \, b_\ell \, z^{-(\ell+1)} \, z^{-(j+1)}$$

$$= z \left(\sum_{j=0}^{\infty} a_j \, z^{-(j+1)} \right) \left(\sum_{\ell=0}^{\infty} b_\ell \, z^{-(\ell+1)} \right) = z \, \hat{a}(z) \, \hat{b}(z),$$

where we have taken $\ell = k - j$. It should be noted that the sequence $\{\delta_k\}$ defined by $\delta_0 = 1$, $\delta_k = 0$, $k > 0$, for which we have seen earlier that $(\mathcal{Z}\{\delta_k\})(z) = \frac{1}{z}$, is the identity for this convolution product, confirmed by

$$(\mathcal{Z}\{\{\delta_k\} * \{a_k\}\})(z) = z \, \frac{1}{z} \, \hat{a}(z) = \hat{a}(z).$$